Separation Axioms on Fuzzy Bitopological Ordered Spaces

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Abstract
In this paper we introduce and study the concept of fuzzy bitopological ordered space $(X, T_1, T_2, \leq)$, which is a fuzzy bitopological space (X, T_1, T_2) with some crisp order$\leq$ on X .Its various properties are analyzed. We also develop and study order separation axioms called pairwise Ti separation axioms for fuzzy bitopological ordered spaces. The relationships between some of these pairwise Ti separa- tion axioms are investigated.

Keywords: fuzzy topology, fuzzy topological ordered space, fuzzy bitopology, separation axioms 2020 **Mathematics Subject Classification:** 54A40, 03E72, 03E72, 54F05

1. Introduction

L. Nachbin in his famous book 'Topology and Order ' published in 1965 [9] studied the relationship between topological and ordered structures. To study of the interdependence between fuzzy topology and order, Katsaras [7] inroduced fuzzy topological ordered spaces in 1981.Here fuzzy topological ordered space is a triplet $(X, T, \leq)$ where T is a fuzzy topology on X and $\leq$ is a crisp order on X. Various separation axioms on fuzzy topological ordered spaces were studied by Bakier and Saady [13],Chaudhari and Das [12], Nikumbh [14][15]. J.C.Kelley [1] introduced bitopological spaces in 1963,since then the bitopological spaces have been subject of intensive investigation for many topologists. Fuzzy bitopological spaces were introduced and studied by A. Kandil et.al. [5]. Bitpological ordered spaces were given by M, K, Singal and A. Singal [4] in 1971. In this paper we introduce Fuzzy Bitopological Ordered Spaces. We develop various separation axioms for fuzzy bitopological ordered spaces. Here we followed Chang's definition of fuzzy topology while ordering used on the space is crisp

2. Preliminaries

Let X be a nonempty set and I be the closed interval [0, 1] with its natural order.

Note: I is a complete completely distributive lattice with order reversing involution defined by $a' = 1 - a, a \in I$.

Definition 2.1: A fuzzy set μ on X is a function on X into I.

Let I^X denote the collection of all fuzzy sets on X.

Definition 2.2: A fuzzy topology *T* on X is a collection of subsets of

I^X such that

1. $\underline{0}, \underline{1} \in T$

2. if $\lambda, \mu \in T$ then $\lambda \wedge \mu \in T$

3. if $\lambda_i \in T$ for all $i \in \Delta$ then $\vee\lambda_i \in T$

(X, T) is called a fuzzy topological space. Members of T are called fuzzy open sets and complements of fuzzy open sets are called fuzzy closed sets.

Definition 2.3: Let X be a nonempty set. A fuzzy topological ordered space is a triple $(X, T, \leq)$ where T is a fuzzy topology on X and $\leq$ is a partial order on X.

Definition 2.4: A fuzzy set μ in a fuzzy topological space (X, T) is called a neighborhood of a point $x \in X$ if there exists an fuzzy open set μ_1 with $\mu_1 \leq \mu$ and $\mu_1(x) = \mu(x) > 0$.

A fuzzy set μ is open if and only if μ is a neighborhood of each $x \in X$ for which $\mu(x) > 0$.

Definition 2.5: A function f from a fuzzy topological space(X, T) to a fuzzy topological space (Y, S) is called fuzzy continuous if $f^{-1}(\mu)$ is open in X for each open set μ in Y .

The function f is continuous at some $x \in X$ if $f^{-1}(\mu)$ is a neighborhood of x for each neighborhood μ of f (x).

f is continuous on X iff f is continuous at each $x \in X$.

Definition 2.6: A fuzzy set μ on an ordered set X is called

i) increasing if $x \leq y \Rightarrow \mu(x) \leq \mu(y)$

ii) decreasing if $x \leq y \Rightarrow \mu(x) \geq \mu(y)$

iii) order convex if $x \leq z \leq y \Rightarrow \mu(z) \geq \mu(x) \wedge \mu(y)$

Note that : Constant functions are increasing, decreasing and order convex. If μ is increasing then $1 - \mu$ is decreasing.

$\{\mu_i \mid i \in \Delta\}$ is increasing (resp. decreasing) then $\mu = \wedge\{\mu_i \mid i \in \Delta\}$is also increasing (resp. decreasing).

Definition 2.7: Let μ be a fuzzy set in a ordered set X then the smallest increasing set containing μ, smallest decreasing set containing μ and smallest convex set containing μ will be denoted by i(μ), d(μ) and c(μ) respectively. Katsaras shown that

i) $i(\mu)(x) = \bigvee\{\mu(y) \mid y \le x\}$. ii) $d(\mu)(x) = \bigvee\{\mu(y) \mid y \ge x\}$.

iii) $c(\mu)(x) = \bigvee\{\mu(x_1) \wedge \mu(x_2) \mid x_1 \le x \le x_2\}$

Note that: $c(\mu) = i(\mu) \wedge d(\mu)$.

The smallest increasing closed set containing μ, the smallest

decreasing closed set containing μ and the smallest convex closed set containing μ will be denoted by I(μ), D(μ) and C(μ) respectively. **Definition 2.8:** Let λ be a fuzzy set of X and μ be a fuzzy set of Y then λ × μ is fuzzy set of X × Y ,defined as

$(\lambda \times \mu)(x, y) = \lambda(x) \wedge \mu(y)$ for each $(x, y) \in X \times Y$

Definition 2.9: Let X, Y be ordered sets and f be a function from

X to Y .Then f is called increasing (resp. decreasing) if $x \le y$ in X

implies $f(x) \le f(y)$(respectively $f(y) \le f(x)$).

Proposition 2.1: Let X, Y be ordered fuzzy topological spaces .A function f: X → Y is increasing if and only if $f^{-1}(\mu)$ is increasing in X for every increasing set μ in Y .

Proof: First, suppose that, f: X → Y is increasing and μ is

increasing set in Y. Let $x \le y$ in X .Then, $f^{-1}(\mu(x)) = \mu(f(x))$, $f^{-1}(\mu(y)) = \mu(f(y))$. Since f is increasing, $\mu(f(x)) \le \mu(f(y))$. As μ is increasing, we have $\mu(f(x)) \le \mu(f(y))$.

∴ $f^{-1}(\mu)(x) \le f^{-1}(\mu)(y)$.So,$f^{-1}(\mu)$ is an increasing function. Similarly,

we can prove the converse.

2.1 Fuzzy Bitopology

Definition 2.10: Let X be a nonempty set, T_1, T_2 be fuzzy topologies on X .Then,(X, T_1, T_2) is called a fuzzy bitopological space.

A topological property can be generalized to the bitopological setting in several ways:

Let (X, T_1, T_2) be a bitopological space and $T = T_1 \vee T_2$.

For a topological property P, we say (X, T_1, T_2) is

- bi-P if both (X, T_1) and (X, T_2) are P .

- join-P if (X, T) is P .

- (1, 2)P if P for T_1 w.r.to T_2.

- (2, 1)P if P for T_2 w.r.to T_1.

- pP (pairwise) if (1, 2)P ∧ (2, 1)P

Definition 2.11: (X, T_1, T_2) be a fuzzy bitopological space. Then

i) (X, T_1, T_2) is p − T_1 iff X is bi − T_1.

ii) (X, T_1, T_2) is p − T_2 if for each pair of distinct points x, y ∈ X there exists a T_1 open set λ and a T_2 open set μ such that λ∧μ = 0.

iii) (X, T_1, T_2) is (i, j) regular if for each point x ∈ X and each i-closed set ν such that ν(x) = 0 there exists an i-open set λ and j-open set μ such that x ∈ λ, ν ≤ μ and λ ∧ μ = 0.

iv) (X, T_1, T_2) is p-normal if for every pair of fuzzy sets λ, μ in X such that λ ∧ μ = 0 ,where λ is T_1closed and μ is T_2 closed there exists a T_2 open set ν such that λ ≤ ν and a T_1 open set δ such that μ ≤ δ and ν ∧ δ = 0.

v) (X, T_1, T_2) is hereditary p-normal if its every bitopological sub- space is p-normal.

Note That: If (X, T , ≤) is a fuzzy topological ordered space,then we can show that

T_u = {μ ∈ T | μ is increasing}

and

T_l = {μ ∈ T | μ is decreasing}

are fuzzy topologies for X ,called respectively the upper and lower fuzzy topologies.

Then, (X, T_u, T_l) is a fuzzy bitopological space.

3. Pairwise T_1 Ordered Space

Definition 3.1: Let X be a nonempty set. A fuzzy bitopological ordered space is a quadruple (X, T_1, T_2 ≤) where T_1, T_2 are fuzzy topologies on X and ≤ is a partial order on X.

Definition 3.2: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is said to be lower (resp. upper) pairwise T_1-ordered if for each x, y ∈ X such that x ≰ y, there exists a decreasing (resp. increasing) T_1 or T_2 neighborhood λ of y (resp. of x) such that x ∉ λ (resp. y ∉ λ),

(X, T_1, T_2, ≤) is said to be pairwise T_1 ordered if it is both upper and lower pairwise T_1 ordered.

Definition 3.3: (X, T_1, T_2, ≤) is said to be pairwise T_0 ordered if it is either upper or lower pairwise T_1 ordered.

Lemma 3.1: If (X, T_1, T_2, and ≤) is pairwise T_1 ordered space then it is pairwise T_0 ordered space

Proof: Follows from the definition.

Proposition 3.1: (X, T_1, T_2, ≤) is a pairwise T_1 ordered space if for each x ∈ X, χ_{lx} and χ_{ux} are T_i closed sets.

Proof: Suppose (X, T_1, T_2, ≤) is a fuzzy bitopological ordered space such that for each x ∈ X, χ_{l_x} and χ_ux are T_i fuzzy closed sets. Let a, b ∈ X with a ≰ b. By hypothesis, λ = χ_{X-lb} ∈ T and λ(a) > 0, λ(b) = 0. Now let x ≤ y we want to show λ(x) ≤ λ(y).

If λ(x) = 0 then the result is obvious.

If λ(x) > 0 then λ(x) = 1. So, x ∈ X − l_b ∴ x ≰ b which imply y ≰ b (because if y ≤ b then x ≤ y ⇒ x ≤ b, which is a contradiction) Hence y ∈ X − l_b that is λ(y) = 1.

∴ λ(x) = λ(y).

So, λ is an increasing neighborhood of a such that λ(a) > 0, λ(b) = The other case may be treated similarly taking μ = $\chi_{(X-ua)}$. Conversely,(X,T_1,T_2,≤) is fuzzy pairwise T_1 ordered. For each pair a ≰ b in X , there exists a decreasing T_i-open neighborhood μ of b such that μ(a) = 0.

For each b ∈ X − u_a,we have χ_{X-u}a (b) > 0 So, μ ≤ χ_{X-u}a .

Hence, χ_{X-ua} is a neighborhood of b so that χ_{ua} is closed.

Similarly, χ_{lb} is closed for each a ∈ X.

Proposition 3.2: (X, T_1, T_2, ≤) is a pairwise lower (resp. upper) T_1 ordered space if for each a, b ∈ X such that a ≰ b,there exists a fuzzy T_i open set λ containing a(resp. b)such that x ≰ b(resp .a ≰ x) for all x ∈ λ.

Proof: Suppose (X, T_1, T_2, ≤) is a pairwise lower T_1 ordered space and a, b ∈ X such that a ≰ b . Then by definition there exists an increasing fuzzy T_1 open set λ such that λ(a) > 0 and λ(b) = 0. For each x ∈ λ we have λ(x) > 0.If x ≤ b then we have λ(b) > λ(x) > 0,which is a contraction. Hence, x ≰ b.

Conversely, suppose for each a ∈ X consider 1 − χ_{ua} .Then b ∈ 1 − χ_{ua} implies a ≰ b. Then by hypothesis there exists a fuzzy open set λ containing a such that x ≰ b for all x ∈ λ. So, λ is an open set such that a ∈ λ ≤ 1 − χ_{ua}. Hence, χ_{ua} is a closed set. Hence, (X, T_1, T_2, ≤) is a pairwise lower T_1 ordered space

Proposition 3.3: If (X, T_1, T_2, ≤) is a pairwise lower (resp. upper) T_1 ordered space and $T_i \le T_i$* for i=1,2) ,then (X, T_1 *, T_2 *, ≤) is also pairwise lower (resp. upper) T_1 ordered.

Proof: Let (X, T_1, T_2, ≤) be a lower pairwise T_1 ordered space. Then for every x, y ∈ X such that x ≰ y, there exists a decreasing T_1 (or T_2) neighborhood λ of y which does not contain x.

Since $T_1 \le T_1$ * ($T_2 \le T_2$ *), there is a decreasing T^* neighborhood (resp. T * neighborhood) of y which do not contain x. Hence X is lower pairwise T_1 ordered space.

Similar, proof for upper pairwise T_1 ordered space.

Theorem 3.1: If (Y, S_1, S_2, $\le'$) is a pairwise T_1 ordered space, f:(X, T_1, T_2, ≤) → (Y, S_1, S_2, $\le'$) is order preserving continuous function. Then (X, T_1, T_2, ≤) is pairwise T_1 ordered space.

Proof: Let x ≤ y in X. Since f is order preserving f (x) $\le'$ f (y) in Y .Hence, there exists an increasing (decreasing) T_i neighborhood λ^* such that λ^*(f (x)) > 0 (λ^*(f (y)) > 0) and λ^*f (y) = 0(λ^*f (x) = 0). Let $\lambda = f^{-1}(\lambda^*)$ .As f is order preserving and fuzzy continuous λ is an increasing (decreasing) T_i neighborhood in X. Also, λ(x) > 0(λ(y) > 0) and λ is not a fuzzy T_i neighborhood of y (resp x). Thus, X is fuzzy T_1 ordered.

Definition 3.4: Let {(X_t, T_{1t}, T_{2t}, $\le_t$) | t ∈ Δ} be a family of or- dered fuzzy bitopological spaces. Let X = $\prod\{Xt \mid t \in \Delta\}$ and let T_1 and T_2 be the product fuzzy topologies on X. Let ≤ ⊂ X × X be defined as, for x = (x_t) and y = (y_t) ∈ X, x ≤ y iff $x_t \le_t y_t$ for all t ∈ Δ .Then, ≤ is a partial order on X. The ordered fuzzy bitopological space (X, T_1, T_2, ≤) is called the ordered fuzzy topological product of the family (X_t, T_1t, T_2t, $\le_t$) | t ∈ Δ}.

Theorem 3.2: The product of a family of pairwise T_1 ordered fuzzy bitopological spaces is a pairwise T_1 ordered fuzzy bitopological space.

Proof: Let {(X_t, T_{1t}, T_{2t}, $\le_t$) | t ∈ Δ} be a family of fuzzy pairwise T_1 ordered fuzzy bitopological spaces and (X, T_1, T_2, ≤) be the fuzzy bitopological product ordered space. Let x =(x_t), y =(y_t) ∈ X be such that x ≰ y .Then, there exists α ∈ Δ such that $x_\alpha \not\le y_\alpha$. Since ($X_\alpha$, $T_{1\alpha}$, $T_{2\alpha}$, $\le_\alpha$) is fuzzy pairwise T_1 ordered ,there exists an increasing open set λ_α in $T_{i\alpha}$ such that $\lambda_\alpha(x_\alpha) > 0$ and $\lambda_\alpha(y_\alpha) = 0$ and an decreasing open set μ_α in $T_i\alpha$ such that $\mu_\alpha(x_\alpha) = 0$ and $\mu_\alpha(y_\alpha) > 0$.

Define $\lambda_t = \prod \{\lambda_t | t \in |\Delta\}$ where $\lambda_t = X$ if $t \neq \alpha$ and

$\mu_t = \prod \{\mu_t | t \in |\Delta\}$ where $\mu_t = X$ if $t \neq \alpha$

Then λ increasing or decreasing T_i open set such that $\lambda(x) > 0$, $\lambda(y)$ = 0 while $\mu(x) = 0$, $\mu(y) > 0$.

$\lambda(y) = \prod\{\lambda t \mid t \in \Delta\}(y)$

$= \min \{\lambda_t(y_t) \mid t \in \Delta\}$

$= \min \{\{\lambda_t(y_t) \mid t \neq \alpha\}, \lambda_\alpha(y_\alpha)\}$

$= \min \{1, 0\}$

$= 0$

Hence, $(X, T_1, T_2, \leq)$ is pairwise T_1 ordered fuzzy bitopological space.

Definition 3.5: Let $(X, T_1, T_2, \leq)$ be a fuzzy bitopological ordered space and $Y \subset X$. Then $(Y, T_{1Y}, T_{2Y}, \leq_Y)$ where $\leq_Y = \leq \cap (Y \times Y)$ and $T_Y = \{\alpha|_Y \mid \alpha \in T_i\}$ is called T_i compatible subspace of $(X, T_1, T_2, \leq)$ iff for each T_{iY} open increasing (resp. decreasing) fuzzy set μ, there exists a T_i open increasing (resp. decreasing) fuzzy set μ^* such that $\mu = \mu^*|_Y$.

Theorem 3.3: Every T_i compatible subspace of a fuzzy pairwise T_1 ordered bitopological space is a fuzzy pairwise T_1 ordered bitopological space.

Proof: Let $(Y, T_{1Y}, T_{2Y}, \leq_Y)$ be a T_i compatible subspace of $(X, T_1, T_2, \leq)$. Let $a, b \in Y$ such that $a \nleq b$. So, $a, b \in X$ such that $a \nleq b$. As X is fuzzy pairwise T_1-ordered there exists an increasing neighborhood λ^* of a in X such that $\lambda^*(b) = 0$ and a decreasing neighborhood μ^* of b in X such that $\mu^*(a) = 0$. Then, $\lambda = \lambda^*|_Y$ is an increasing neighborhood of a in Y such that $\lambda(b) = 0$ and $\mu = \mu^*|_Y$ is a decreasing neighborhood of b in Y such that μ(a) = 0.

Hence (Y, T_{1Y}, T_{2Y}, $\leq_Y$) is fuzzy pairwise T_1 ordered.

4. Pairwise T_2 ordered space

Definition 4.1: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is said to be pairwise T_2-ordered if for each x, y ∈ X such that x ≰ y, there exists a decreasing T_1 neighborhood (or a T_2 neighborhood) λ of y and a increasing T_2 neighborhood (or resp. a T_1 neighborhood) μ of x such that λ ∧ μ = 0.

Proposition 4.1: Let (X, T_1, T_2, ≤) be a fuzzy pairwise T_2-ordered space and Y ⊂ X. Then, every T -compatible subspace (Y,T_{1Y}, T_{2Y}, $\leq_Y$) is also fuzzy pairwise T_2 ordered.

Proof: Let (Y,T_{1Y}, T_{2Y}, $\leq_Y$) be a T compatible subspace of (X, T_1, T_2, ≤). Let a, b ∈ Y such that a ≰ b. So, a, b ∈ X such that a ≰ b. As X is pairwise T_2-ordered there exists an increasing neighborhood λ^* of a in X and a decreasing neighborhood μ^* of b in X such that $\lambda^* \wedge \mu^* = 0$.Then, $\lambda = \lambda^*|_Y$ is an increasing neighborhood of a in Y and $\mu = \mu^*|_Y$ is a decreasing neighborhood of b in Y such that λ ∧ μ = 0.

Hence (Y, T_{1Y}, T_{2Y}, $\leq_Y$) is fuzzy T_2 ordered.

Proposition 4.2: If f is a order preserving fuzzy continuous mapping from (X, T_1, T_2, ≤) to a fuzzy pairwise T_2ordered space (Y, δ_1, δ_2, $\leq'$) then (X, T_1, T_2, ≤) is also fuzzy pairwise T_2 ordered.

Proof: Suppose f: (X, T_1, T_2, ≤) → (Y, δ_1, δ_2, $\leq'$) is an order preserving fuzzy continuous map. Let x ≰ y in X. Hence f (x) ≰ f (y) in Y. But (Y, δ_1, δ_2, $\leq'$) is a fuzzy T_2 ordered space, so there exists an increasing fuzzy open set λ and a decreasing fuzzy open set μ such that λ is a fuzzy open neighborhood of f (x) and μ is a fuzzy open neighborhood of f (x) such that λ ∧ μ = 0.

Since, f is increasing, λ is increasing it follows that $f^{-1}(\lambda)$ is increasing. Also, since f is increasing, μ is decreasing it follows that $f^{-1}(\mu)$ is decreasing. Also, f is continuous, implies $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are fuzzy open sets containing x and y respectively.

$$f^{-1}(\lambda) \wedge f^{-1}(\mu) = f^{-1}(\lambda \wedge \mu) = f^{-1}(0) = 0$$

Hence, X is fuzzy T_2 ordered. Similarly we can prove the result for decreasing function.

Theorem 4.1: The product of a family of fuzzy pairwise T_2 ordered bitopological spaces is also fuzzy pairwise T_2 ordered bitopological space.

Proof: Let $\{(X_t, T_{1t}, T_{2t}, \leq_t) \mid t \in \Delta\}$ be a family of fuzzy T_2 ordered spaces and $(X, T_1, T_2, \leq)$ be the product of ordered fuzzy bitopological spaces.

If $(x_t, y_t) \in X$ such that $x_t \not\leq y_t$ then there exists $t_0 \in \Delta$ such that $x_{t0} \not\leq y_{t0}$ .Then there exists fuzzy open sets λ_{t0} and μ_{t0} in X_{t0} such that λ_{t0} is increasing and μ_{t0} is decreasing ,λ_{t0} is a fuzzy open neighborhood of x_{t0} , μ_{t0} is fuzzy open neighborhood of y_{t0} and $\lambda_{t0} \wedge \mu_{t0} = 0$.

Define $\lambda = \prod_{t \in \Delta} \lambda_t$ where $\lambda_t = 1_{X_t}$ when $t \neq t_0$

and

$\mu = \prod_{t \in \Delta} \mu_t$ where $\mu_t = 1_{X_t}$ when $t \neq t_0$.

Then λ is an increasing fuzzy open set of X and is a decreasing fuzzy open set of X such that λ is a fuzzy open neighborhood of x_t and μ is a fuzzy open neighborhood of y_t and $\lambda \wedge \mu = 0$.

Hence $(X, T_1, T_2, \leq)$ is fuzzy pairwise T_2 ordered bitopological space.

Proposition 4.3: Every fuzzy pairwise T_2 ordered bitopological space is a fuzzy pairwise T_1 ordered space.

Theorem 4.2: A fuzzy bitopological ordered space $(X, T_1, T_2, \leq)$ is a pairwise T_2 ordered space then for each pair $a, b \in X$ such that $a \not\leq b$, there exists a increasing T_i fuzzy open set λ and a decreasing T_j fuzzy open set μ such that $\lambda(a) > 0$, $\mu(b) > 0$ and $\lambda(x) > 0$, $\mu(y) > 0$, together imply $x \not\leq y$.

Proof: Suppose (X, T_1, T_2, ≤) is a pairwise fuzzy T_2 ordered space.

Let a, b ∈ X such that a ≰ b, there exists a increasing T_i fuzzy open set λ and a decreasing T_j fuzzy open set μ such that λ(a) > 0, μ(b) > 0 and

λ(x) > 0, μ(y) > 0.

Suppose x ≤ y. As λ is increasing and μ is decreasing we have λ(x) ≤ λ(y) and μ(y) ≤ μ(x),we get 0 < λ(x) ∧ μ(y) ≤ λ(y) ∧ μ(x), which is a contradiction because λ ∧ μ = 0.

Definition 4.2: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is called fuzzy almost pairwise T_2 ordered space if for any a, b ∈ X such that a ≰ b ,there exists a increasing T_i fuzzy open set λ and a decreasing T_j fuzzy open set μ such that λ(a) > 0, μ(b) > 0 and λ(x) > 0, μ(y) > 0, together imply x ≰ y

Remark 4.1: Above theorem shows that if (X, T_1, T_2, ≤) is a fuzzy pairwise T_2 ordered space then (X, T_1, T_2, ≤) is a fuzzy almost pairwise T_2 ordered space.

Definition 4.3: A fuzzy bitopoogical ordered space is said to be pair- wise Urysohn ordered space if for x, y ∈ X such that x ≰ y ,there exists an increasing T_i open set λ and a decreasing T_j open set μ such that λ(x) > 0, μ(y) > 0 and I_j(λ) ∧ D_i(μ) = 0, i ≠ j, i, j = 1, 2.

Clearly, every fuzzy pairwise Urysohn ordered bitopological space is a fuzzy pairwise T_2 ordered space.

5. Pairwise Regular Ordered Space

Definition 5.1: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is said to be pairwise lower (resp. upper)regular ordered if for each T_i- closed decreasing fuzzy set λ (resp.T_i- closed increasing fuzzy set) and for each x ∈ X such that λ(x) = 0 , there exists an increasing T_j open set μ (resp. decreasing T_j open set) and a decreasing T_i open set ν (resp. increasing T_i open set) such that μ(x) > 0, λ ≤ ν, μ ∧ ν = 0.

(X, T_1, T_2, ≤) is said to be pairwise regular ordered iff it is both pairwise upper and lower regular ordered.

Definition 5.2: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is said to be pairwise T_3 ordered if it is pairwise regular ordered and pairwise T_1 ordered.

Proposition 5.1: A fuzzy bitopological ordered space (X, T_1, T_2, ≤)

is fuzzy pairwise lower (resp. upper) regularly ordered iff the following condition holds: For each x ∈ X and an increasing (resp. decreasing)T_i- open fuzzy neighborhood μ of x, there exists an increasing (resp. decreasing) T_j open set ν such that ν(x) > 0 and $\nu \leq I_j(\nu) \leq \mu$(resp. $\nu \leq D_j(\nu) \leq \mu$).

Proof: Suppose (X, T , ≤) is fuzzy lower (resp. upper) regular ordered space. Let x ∈ X and let μ be an increasing (resp .decreasing)T_i-open neighborhood of x, then 1 − μ is T_i-closed, decreasing(increasing)in X and (1 − μ)(x) = 0.

By hypothesis there exists increasing (decreasing) fuzzy T_j open set ν and decreasing (increasing) fuzzy T_i open set λ such that ν(x) > 0,

$1-\mu \leq \lambda$, $\lambda \wedge \nu = 0$.Hence,$\nu \leq 1-\lambda \leq \mu$ So, $I_j(\nu) \leq I_i(1-\lambda) = 1-\lambda$. Since

$1 - \lambda$ is T_i-closed, $\nu \leq I_j(\nu) \leq \mu(\nu \leq D_j(\mu) \leq \mu)$

Converse is straightforward.

Proposition 5.2: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is fuzzy pairwise regular ordered if and only if for every x ∈ X and decreasing(resp. increasing) T_i-open fuzzy set λ containing x, there exists a decreasing (resp. increasing) T_i-open fuzzy set μ and a decreasing (resp. increasing) T_j-closed fuzzy set ν such that $\mu \leq \nu \leq \lambda$.

Proof: Follows from the above proposition.

Proposition 5.3: If (X, T_1, T_2, ≤) is fuzzy pairwise regular ordered space then every T-compatible ordered subspace (Y, T_Y, $\leq_Y$) is also fuzzy regularly ordered.

Proof: Let (Y, T_{1Y} , T_{2Y} , $\leq_Y$) be T -compatible ordered subspace of the fuzzy pairwise upper regularly ordered space (X, T_1, T_2, ≤)

and let x ∈ Y and μ be any T_{iY} open decreasing fuzzy neighborhood of xin Y .Thus, there exists a T_i-open decreasing fuzzy set λ^* such that $\lambda = \lambda^*|_Y$ with $\lambda^*(x) > 0$.Since,(X, T_1, T_2, , ≤) is fuzzy pairwise upper regular ordered then there exists a T_i-open decreasing fuzzy set μ^* such that $\mu^*(x) > 0$ and $\mu^* \leq D(\mu^*) \leq \lambda^*$

By restriction of μ^* and $D(\mu^*)$ by Y we have that $\mu \leq D(\mu) \leq \lambda$ and

(Y, T_{1Y} , T_{2Y} , $\leq_Y$) is fuzzy pairwise upper regular ordered.

Proposition 5.4: If (X, T_1, T_2, ≤) is fuzzy pairwise lower or upper T_3 ordered then (X, T_1, T_2, ≤) is fuzzy pairwise T_2 ordered.

Proof: Let x ≰ y in X and suppose (X, T_1, T_2, ≤) is pairwise lower T_3-ordered then χ_{l_y} is T_i-closed and decreasing and χ_{l_y} (x) = 0. Since, (X, T_1,

T_2, ≤) is fuzzy pairwise regularly ordered, there exists an increasing T_i neighborhood μ of x and a decreasing T_j neighborhood ν of l_y such that $\mu \wedge \nu = 0$.

Since, ν is a T_j-neighborhood of l_y, it is a T_j-neighborhood of y. So, (X, T_1, T_2, ≤) is fuzzy pairwise T_2 ordered.

Theorem 5.1: The product of a family of fuzzy pairwise regular ordered spaces is also fuzzy pairwise regular ordered.

Proof: Let $\{(X_t, T_{1t}, T_{2t}, \leq_t) \mid t \in \Delta\}$ be a family of fuzzy pairwise regular ordered spaces and (X, T_1, T_2, ≤) be the product of fuzzy topological ordered spaces.

Consider, x ∈ X in the product topology.

Let μ be a decreasing fuzzy T_i-open set containing x. Since, the projection $P_\alpha : X \to X_t$ is order preserving continuous function, the point x_t is contained in a decreasing T_it -open set λ_t for each t ∈ Δ such that $\mu = \{P^{-1}(\lambda_t) \mid t \in \Delta\}$.

As $(X_t, T_{1t}, T_{2t}, \leq_t)$ is fuzzy pairwise regular ordered, there exists a decreasing T_{jt} -open set ν_t such that

$x_t \in \nu_t \leq D(\nu_t) \leq \mu_t$

$x \in P_t^{-1}(\nu_t) \leq P_t^{-1}(D(\nu_t)) \leq \mu$

Hence, (X, T_1, T_2, ≤) is fuzzy pairwise regular ordered.

6. Pairwise Normal Ordered Space

Definition 6.1: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is said to be pairwise normal ordered if for each decreasing fuzzy T_i closed set λ_1 and each increasing fuzzy T_j closed set λ_2 such that $\lambda_1 \wedge \lambda_2 = 0$, there exist fuzzy T_j open decreasing set μ_1 and fuzzy T_i open increasing set μ_2 such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \wedge \mu_2 = 0$.

Definition 6.2: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is said to be pairwise T_4 ordered if it is pairwise normal ordered and pairwise T_1 ordered.

Definition 6.3: A fuzzy bitopological ordered space (X, T_1, T_2, ≤) is called a fuzzy pairwise normally ordered space iff the following condi- tion is satisfied : Given a decreasing(resp. increasing) T_i-closed fuzzy set μ and a decreasing (resp. increasing) T_j-open fuzzy set ρ such that$\mu \leq \rho$, there exists a decreasing (resp. increasing) T_i-open fuzzy set ρ_1 and a decreasing(resp. increasing)T_j-closed fuzzy set μ_1 such that $\mu \leq \rho_1 \leq \mu_1 \leq \rho$.

Definition 6.4: If μ is a fuzzy set in a fuzzy bitopological ordered space X, we define

$D_{T_i}(\mu) = \inf\{\rho \mid \rho \geq \mu, T_i$ closed and decreasing$\}$ Clearly, $D_{T_i}(\mu)$ is the smallest decreasing fuzzy set in $(X, T_i, \leq)$ which contains μ.

Proposition 6.1: $(X, T_1, T_2, \leq)$ is a fuzzy pairwise normally ordered space iff the following condition is satisfied:

Given a decreasing (resp. increasing) Ti closed fuzzy set μ and a de- creasing (resp. increasing) T_j-open fuzzy set ρ with $\mu \leq \rho$, there exists a decreasing (resp. increasing) T_j open fuzzy set ρ_1 such that $\mu \leq \rho_1 \leq D_i(\rho_1) \leq \rho$ (resp.$\mu \leq \rho_1 \leq I_i(\rho_1) \leq \rho$).

Proof: Let (X, T_1, T_2, ρ) be a fuzzy pairwise normally ordered space. Let μ, ρ be given as in proposition. By definition, we have a decreasing fuzzy T_j open set $\rho_1$and a decreasing fuzzy T_i closed set μ_1 such that $\mu \leq \rho_1 \leq \mu_1 \leq \rho$.

Since, μ_1 is a decreasing fuzzy T_i closed set such that $\rho_1 \leq \mu_1$ we have

$\mu \leq \rho_1 \leq D_i(\rho_1) \leq \mu_1 \leq \rho$.

Conversely, suppose μ is a decreasing fuzzy T_i closed set and ρ is a decreasing fuzzy T_j open set such that $\mu \leq \rho$. Hence by condition of proposition, there exists a decreasing fuzzy T_j open set ρ_1 such that

$\mu \leq \rho_1 \leq D_i(\rho_1) \leq \rho$. Clearly, $D_i(\rho_1)$ is the smallest decreasing fuzzy T_i closed set containing ρ_1.Put $\mu_1 = D_i(\rho_1)$. Then, $\mu \leq \rho_1 \leq \mu_1 \leq \rho$. Hence, (X, T_1, T_2, ρ) is a fuzzy pairwise normally ordered space.

Proposition6.2: Every fuzzy pairwise normally ordered bitopological space is fuzzy pairwise regularly ordered space.

Proof: Suppose $(X, T_1, T_2, \leq)$ be a normally ordered space. Let $x \in X$,

μ be a decreasing T_i-closed fuzzy set and ρ be a decreasing T_j-open neighborhood of x with $\mu \leq \rho$. By normality, there exists a decreasing T_j-open fuzzy set λ such that $\mu \leq \lambda \leq D_i(\lambda) \leq \rho$. So, $(x, T_1, T_2, \leq)$ is fuzzy regularly ordered.

Definition 6.5: A fuzzy pairwise normally ordered bitopological space which is also fuzzy pairwise T_1 ordered is called fuzzy pairwise T_4 ordered bitopological space.

Corollary 6.1: Every fuzzy pairwise T_4 ordered bitopological space is fuzzy pairwise T_3 ordered bitopological space.

Proof: follows from proposition

Proposition 6.3: Every biclosed subspace of a fuzzy pairwise normally ordered bitopological space is fuzzy pairwise normally ordered bitopological space.

Proof: Let (Y, T_{1Y},T_{2Y} ,$\leq_Y$) be a biclosed subspace of a fuzzy pairwise normal ordered space (X, T_1, T_2, ≤).

Let μ^* and ρ^* be a decreasing T_{1Y} -closed and T_{2Y} -open fuzzy sets respectively

such that $\mu^* \leq \rho^*$.

Since, μ^* and $1 - \rho^*$ are T_{1Y} -closed and T_{2Y} -closed fuzzy sets respectively and Y is closed, so,μ^* and $1 - \rho^*$ are T_1-closed and T_2-closed fuzzy sets.

∴ μ^* and ρ^* are T_1-closed and T_2-open fuzzy sets respectively with

$\mu^* \leq \rho^*$.

Since (X, T , ≤) is fuzzy normal ordered there exists a decreasing T_i-open fuzzy set ρ_1 and a decreasing T_j-closed fuzzy set μ_1 such that

$\mu^* \leq \rho_1 \leq \mu_1 \leq \rho^*$

It follows that, $\rho^* = \rho_1|_Y$ is a decreasing T_{1Y} open fuzzy set and $\mu^* = \mu_1|_Y$

is a T_{2Y} -closed fuzzy set such that $\rho^* \leq \mu^*$ and $\mu_1^* \leq \rho_1^* \leq \mu^* \leq \rho*$. 1 1

Hence, (Y, T_{1Y} ,T_{2Y},$\leq_Y$) is fuzzy pairwise normally ordered.

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